ASYMPTOTIC ENTANGLEMENT OF TWO INDEPENDENT SYSTEMS IN A COMMON BATH

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Two, non-interacting systems immersed in a common bath and evolving with a Markovian, completely positive dynamics can become initially entangled via a purely noisy mechanism. Remarkably, for certain, phenomenologically relevant environments, the quantum correlations can persist even in the asymptotic long-time regime.

1. Introduction

The simplest way to generate quantum correlations between two systems is via a suitable Hamiltonian coupling: optimization of entanglement production can be achieved by a careful choice of the form of the interaction term.[1−5]

When the two systems are immersed in an external bath, decoherence phenomena usually occur, counteracting entanglement generation. These effects are clearly a curse in quantum information and in fact various error correcting strategies have been devised in order to limit these environment induced mixing enhancing phenomena.

However, an external environment can also provide a further, indirect coupling between the two systems and therefore an additional mechanism to correlate them.[6−9] That the external environment can indeed generate entanglement has been first established in exactly solvable models:[6] there, correlations between the two subsystems take place during a short time transient phase, where the reduced dynamics of the subsystems contains memory effects.

Remarkably, a similar phenomenon of entanglement production may occur also in the Markovian regime, through a purely noisy mechanism.[10−12,28,29] For the case of two, two-level systems immersed in a common bath this phenomenon can be established by looking at the eigenvalues of the partial transposed density matrix

that represents the two subsystem state.[13,14] A sufficient condition for initial, environment induced entanglement generation can then be obtained: it allows deriving a test on the entanglement power of the bath.[10]

Nevertheless, this test is unable to determine the fate of the initially created quantum correlations as time becomes large. In order to discuss asymptotic entanglement, one has to analyze directly the structure of the dissipative, Markovian dynamics followed by the two systems, and determine its equilibrium states.

In the following, we shall present such an investigation for a subsystem composed by a couple of identical two-level systems, evolving with a completely positive dynamical semigroup. Being interested in discussing the correlation power of the environment, we shall assume the two systems to be independent, without any mutual direct interaction.[a] As we shall see, there exists a class of environments, which is of relevance in phenomenological applications, that are able not only to initially generate entanglement: they can continue to enhance it even in the asymptotic long time regime.

2. Master Equation

We shall deal with a subsystem composed by two, identical, non-interacting two-level systems, immersed in a common heat bath. The derivation of a physically consistent master equation for the reduced density matrix $\rho(t) \equiv \mathrm{Tr}_{\mathcal{E}}[\rho_{\rm tot}(t)]$, obtained by tracing the total density matrix $\rho_{\rm tot}(t)$ over the bath degrees of freedom, is notoriously tricky, requiring an *a priori* unambiguous separation between subsystem and environment.[15−18,28] Generally speaking, this distinction can be achieved when the correlations in the environment decay much faster than the characteristic evolution time of the subsystem alone. Then, in the limit of weak couplings, the changes in the evolution of the subsystem occur on time scales that are very long, so large that the details of the internal environment dynamics result irrelevant.

This situation is amenable to a precise mathematical treatment:[19] as a result, the two-system state $\rho(t)$ evolves in time according to a quantum dynamical semigroup of completely positive maps, generated by a master equation in Kossakowski-Lindblad form:[19,20,21]

$$\frac{\partial\rho(t)}{\partial t} = -i\big[\mathcal{H}_{\rm eff},\, \rho(t)\big] + \mathcal{L}[\rho(t)]\ . \tag{1}$$

The unitary term depends on an effective Hamiltonian, containing both the initial system Hamiltonian and suitable Lamb contributions; in general, it can be decomposed as: $\mathcal{H}_{\rm eff} = H_{\rm eff}^{(1)} + H_{\rm eff}^{(2)} + H_{\rm eff}^{(12)}$. The first two terms represent single system contributions,

$$H_{\rm eff}^{(1)} = \sum_{i=1}^{3} H_i^{(1)}(\sigma_i \otimes \mathbf{1})\ , \quad H_{\rm eff}^{(2)} = \sum_{i=1}^{3} H_i^{(2)}(\mathbf{1} \otimes \sigma_i)\ , \tag{2}$$

[a] In the same way, we shall also neglect any Lamb shift Hamiltonian contribution that might be generated by the presence of the external bath.

where $(\sigma_i \otimes \mathbf{1})$, $(\mathbf{1} \otimes \sigma_i)$ are the basis operators pertaining to the two systems, respectively, with σ_i, $i = 1, 2, 3$ the Pauli matrices; the third piece is a bath generated direct two-system coupling term, that can be expressed as:

$$H_{\rm eff}^{(12)} = \sum_{i,j=1}^{3} H_{ij}^{(12)} \, (\sigma_i \otimes \sigma_j) \ . \tag{3}$$

The dissipative contribution $\mathcal{L}[\rho(t)]$ can be cast in Kossakowski form,[20]

$$\mathcal{L}[\rho] = \sum_{\alpha,\beta=1}^{6} C_{\alpha\beta} \Big[\mathcal{F}_\beta \, \rho \, \mathcal{F}_\alpha \, - \, \frac{1}{2} \Big\{ \mathcal{F}_\alpha \mathcal{F}_\beta \, , \, \rho \Big\} \Big] \ , \tag{4}$$

using the hermitian, traceless, matrices $\mathcal{F}_\alpha$ that coincide with the first system basis operators $(\sigma_\alpha \otimes \mathbf{1})$ for $\alpha = 1, 2, 3$, while reproducing the second system basis operators $(\mathbf{1} \otimes \sigma_{\alpha-3})$ for $\alpha = 4, 5, 6$. The Kossakowski matrix $C_{\alpha\beta}$ is a 6×6 matrix, which is non-negative, thus guaranteeing the complete positivity of the reduced dynamics. Using the splitting of the indices introduced above, it can be conveniently written as

$$C = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^\dagger & \mathcal{C} \end{pmatrix} \ , \tag{5}$$

in terms of 3×3 matrices $\mathcal{A} = \mathcal{A}^\dagger$, $\mathcal{C} = \mathcal{C}^\dagger$ and $\mathcal{B}$. This decomposition carries a direct physical interpretation. Indeed, the pieces in (4) containing the diagonal contributions $\mathcal{A}$ and $\mathcal{C}$ correspond to noise terms that affect the first, respectively the second, system in absence of the other. On the contrary, the pieces depending on $\mathcal{B}$ encode environment generated dissipative couplings between the two, otherwise independent, systems.

In order to obtain a more explicit expression for the dynamics, it is convenient to decompose the 4×4 density matrix $\rho(t)$ along the Pauli matrices:

$$\rho(t) = \frac{1}{4} \Big[\mathbf{1} \otimes \mathbf{1} + \sum_{i=1}^{3} \rho_{0i}(t) \ \mathbf{1} \otimes \sigma_i + \sum_{i=1}^{3} \rho_{i0}(t) \ \sigma_i \otimes \mathbf{1} + \sum_{i,j=1}^{3} \rho_{ij}(t) \ \sigma_i \otimes \sigma_j \Big] \ , \tag{6}$$

where the coefficients $\rho_{0i}(t)$, $\rho_{i0}(t)$, $\rho_{ij}(t)$ are all real. Substitution of this expansion in the master equation (1) allows deriving the corresponding evolution equations for the above components of $\rho(t)$. As mentioned in the Introduction, we are interested in studying possible entanglement production through the purely dissipative action of the environment; in the following, we shall therefore ignore the Hamiltonian pieces and concentrate on the study of the effects induced by the dissipative part $\mathcal{L}[\,\cdot\,]$ in (4).[b]

[b]In other terms, we limit our analysis to baths for which the induced two-system Hamiltonian coupling in (3) is vanishingly small or alternatively does not give rise to temperature dependent entanglement phenomena. This situation is rather common in phenomenological applications; for concrete examples, see Refs.[11,12].

Although the general case can be similarly treated, for sake of simplicity we shall here limit our considerations to baths for which the submatrices in (5) are all equal: $\mathcal{A}=\mathcal{B}=\mathcal{C}$. This choice of the Kossakowski matrix, although special, is nevertheless of great phenomenological relevance since it is adopted in the analysis of the phenomenon of resonance fluorescence.[22,23] In addition, it is precisely a dissipative term of this type that describes the interaction of two atoms with a set of weakly coupled external quantum fields, in the limit of a vanishing spatial atom separation.[11,12]

In such a situation, the form of the dissipative contribution in (4) simplifies so that the evolution equation can be rewritten as

$$\frac{\partial\rho(t)}{\partial t}=\sum_{i,j=1}^{3}\mathcal{A}_{ij}\Big[\Sigma_j\,\rho(t)\,\Sigma_i-\frac{1}{2}\Big\{\Sigma_i\Sigma_j\,,\,\rho(t)\Big\}\Big]\ , \tag{7}$$

in terms of the following symmetrized two-system operators

$$\Sigma_i=\sigma_i\otimes\mathbf{1}+\mathbf{1}\otimes\sigma_i\ ,\quad i=1,2,3\ . \tag{8}$$

One easily checks that these operators obey the same $su(2)$ Lie algebra of the Pauli matrices; further, together with

$$S_{ij}=\sigma_i\otimes\sigma_j+\sigma_j\otimes\sigma_i\ ,\quad i,j=1,2,3\ , \tag{9}$$

they form a closed algebra under matrix multiplication, whose explicit expression is collected in the Appendix.

It is now convenient to decompose the hermitian matrix $\mathcal{A}_{ij}$ into its real and imaginary parts,

$$\mathcal{A}_{ij}=A_{ij}+i\sum_{k=1}^{3}\varepsilon_{ijk}B_k\ , \tag{10}$$

with A_{ij} real symmetric and B_i real. Inserting this in (7) and using the decomposition (6), a straightforward but lengthy calculation allows derive the following evolution equations for the components of $\rho(t)$:

$$\begin{aligned}
\frac{\partial\rho_{0i}(t)}{\partial t}&=-2A\rho_{0i}(t)+2\sum_{k=1}^{3}\Big[A_{ik}\rho_{0k}(t)-\rho_{ik}(t)B_k\Big]+2(2+\tau)B_i\ ,\\
\frac{\partial\rho_{i0}(t)}{\partial t}&=-2A\rho_{i0}(t)+2\sum_{k=1}^{3}\Big[A_{ik}\rho_{k0}(t)-\rho_{ki}(t)B_k\Big]+2(2+\tau)B_i\ ,\\
\frac{\partial\rho_{ij}(t)}{\partial t}&=-4A\big[\rho_{ij}(t)+\rho_{ji}(t)\big]+2\sum_{k=1}^{3}\Big[A_{ik}\rho_{kj}(t)+A_{jk}\rho_{ik}(t)\Big]-4A_{ij}\tau\\
&\quad+4\sum_{k=1}^{3}\Big[A_{ik}\rho_{jk}(t)+A_{jk}\rho_{ki}(t)\Big]+4\Big[A\tau-\sum_{k,l=1}^{3}A_{kl}\rho_{lk}(t)\Big]\delta_{ij}\\
&\quad+2\Big[B_i\rho_{j0}(t)+B_j\rho_{0i}(t)\Big]+4\Big[B_i\rho_{0j}(t)+B_j\rho_{i0}(t)\Big]
\end{aligned}$$

$$-2\sum_{k=1}^{3} B_k\left[\rho_{0k}(t)+\rho_{k0}(t)\right]\delta_{ij}\ . \tag{11}$$

In these formulas, the parameter A represents the trace of A_{ij} while the quantity $\tau \equiv \sum_{i=1}^{3}\rho_{ii}$ that of the submatrix ρ_{ij}. By taking the trace of both sides of the last equation above, one discovers that τ is a constant of motion.[c] Nevertheless, the value of τ can not be chosen arbitrarily; the requirement of positivity of the initial density matrix $\rho(0)$ readily implies: $-3 \leq \tau \leq 1$.

3. Environment Induced Entanglement Generation

Using the explicit form (11) for the master equation derived in the previous Section, one can now investigate whether an external environment can actually entangle the two independent systems. Since we are dealing with a couple of two-level systems, this can be achieved with the help of the partial transposition criterion:[13,14] a state $\rho(t)$ results entangled at time t if and only if the operation of partial transposition does not preserve its positivity.

We shall first discuss the possibility of entanglement creation at the beginning of the evolution: if a bath is not able to initially entangle the two systems, it will hardly do so in the limit of large times. A simple strategy to ascertain entanglement creation is as follows: assume the initial state to be pure and separable, *i.e.* $\rho(0)=|\varphi\rangle\langle\varphi|\otimes|\psi\rangle\langle\psi|$, and then find out whether in the neighborhood of $t=0$ the dynamics in (11) is able to make negative an initially zero eigenvalue of the partially transposed density matrix $\tilde{\rho}(t)$ (note that $\tilde{\rho}(0)\equiv\rho(0)$ for the chosen initial state). This amounts to study the behavior of the time derivative $\partial_t\tilde{\rho}(0)$, that can be explicitly obtained by taking the partial transposition of both sides of (7) (or equivalently of the system of equations in (11)).

In this way, one finds that the dynamics generated by the equations in (7) can indeed entangle the two subsystems, provided at least one of the coefficients B_i in the Kossakowski matrix $\mathcal{A}_{ij}$ in (10) is nonvanishing (see Ref.[10] for further details). This is a generic property of the Markovian dynamics in (7): entanglement is generated as soon as $t>0$.

In order to study the fate of this initially produced correlations as time becomes large, one needs to analyze the ergodic properties of the semigroup evolution generated by (7). On general grounds, one expects that the effects of decoherence and dissipation that counteract entanglement production be dominant at large times, so that no entanglement is left at the end. However, as we shall explicitly see, there are cases for which the environment induced entanglement creation never stops as time flows, allowing at the end the presence of entangled equilibrium states.

[c] By analyzing the structure of a general evolution equation with the dissipative term as in (4), one can show that this is the case only when the condition $\mathcal{A}=\mathcal{B}=\mathcal{C}$ is satisfied. This result is also related to the existence of multiple equilibrium states; see below.

The system of first order differential equations in (11) naturally splits into two independent sets, involving the symmetric, $\rho_{(0i)} = \rho_{0i} + \rho_{i0}$, $\rho_{(ij)} = \rho_{ij} + \rho_{ji}$, and antisymmetric, $\rho_{[0i]} = \rho_{0i} - \rho_{i0}$, $\rho_{[ij]} = \rho_{ij} - \rho_{ji}$, variables. By examining the structure of the two sets of differential equations, one can conclude that the antisymmetric variables involve exponentially decaying factors, so that they vanish for large times. Then, recalling the definitions in (8) and (9), the study of the equilibrium states $\hat\rho$ of the evolution (7) can be limited to density matrices of the form:

$$\hat\rho = \frac{1}{4}\left[\mathbf{1}\otimes\mathbf{1} + \sum_{i=1}^{3}\hat\rho_i\,\Sigma_i + \sum_{i,j=1}^{3}\hat\rho_{ij}\,S_{ij}\right]\,, \tag{12}$$

with $\hat\rho_{ij} = \hat\rho_{ji}$.

These states obey the equilibrium condition $\partial_t\hat\rho = 0$, and therefore annihilates the r.h.s. of all equations in (11). By direct inspection, one finds that these conditions are invariant under linear orthogonal transformations that act on both the coefficients B_i, A_{ij} and the components $\hat\rho_i$, $\hat\rho_{ij}$ of the density matrix $\hat\rho$. Then, without loss of generality, for the purpose of identifying the asymptotic states, one can take the real part of $\mathcal{A}_{ij}$ to be diagonal, *i.e.* $A_{ij} = \lambda_i\,\delta_{ij}$; the general case can always be recovered at the end by undoing the orthogonal transformation that has brought A_{ij} in diagonal form.

Further, in order to simplify the exposition, we shall assume the vector of components B_i to be directed along the third axis, so that only the component $B_3 \equiv B$ will be nonvanishing. Then, positivity of $\mathcal{A}_{ij}$ readily implies: $\lambda_i \geq 0$, $i = 1,2,3$ and $B^2 \leq \lambda_1\lambda_2$.

The approach to equilibrium of semigroups whose generator is of the generic Kossakowski-Lindblad form has been studied in general and some rigorous mathematical results are available.[24,16] We shall present such results by adapting them to the case of the evolution generated by the equation (7).

First of all, one notice that in the case of a finite dimensional Hilbert space, there always exists at least one stationary state $\hat\rho_0$.[d] Let us now introduce the operators $V_i = \sum_{j=1}^3 \mathcal{A}_{ij}^{1/2}\,\Sigma_j$ (recall that $\mathcal{A}$ is non-negative), so that the r.h.s. of (7) can be rewritten in the so-called diagonal form:

$$\mathcal{L}[\rho] = \sum_{i,j=1}^{3}\left[V_j\,\rho\,V_i^\dagger - \frac{1}{2}\left\{V_i^\dagger V_j\,,\,\rho\right\}\right]\,. \tag{13}$$

When the set $\mathcal{M}$ formed by all operators that commute with the linear span of $\{V_i,\ V_i^\dagger,\ i = 1,2,3\}$ contains only the identity, one can show that the stationary state $\hat\rho_0$ is unique, and of maximal rank. On the other hand, when there are several

[d]This can be understood by recalling that in finite dimensions the ergodic average of the action of a completely positive one-parameter semigroup on any initial state always exists: the result is clearly a stationary state.

stationary states, they can be generated in a canonical way from a $\hat\rho_0$ with maximal rank using the elements of the set $\mathcal{M}$.

In the present case, $\mathcal{M}$ contains the operator $S \equiv \sum_{i=1}^3 S_{ii}$, besides the identity; indeed, with the help of the algebraic relations collected in the Appendix, one immediately finds: $[S,\ \Sigma_i] = 0$. Out of these two elements of $\mathcal{M}$, one can now construct two mutually orthogonal projection operators:[e]

$$P = \frac{1}{4}\left[\mathbf{1}\otimes\mathbf{1} - \frac{S}{2}\right]\,, \qquad Q = 1 - P\ . \tag{14}$$

Then, one can show that any given initial state $\rho(0)$ will be mapped by the evolution (7) into the following equilibrium state:

$$\rho(0) \to \hat\rho = \frac{P\,\hat\rho_0\,P}{\mathrm{Tr}\big[P\,\hat\rho_0\,P\big]}\ \mathrm{Tr}\big[P\,\rho(0)\big] + \frac{Q\,\hat\rho_0\,Q}{\mathrm{Tr}\big[Q\,\hat\rho_0\,Q\big]}\ \mathrm{Tr}\big[Q\,\rho(0)\big]\ . \tag{15}$$

That this state is indeed stationary can be easily proven by recalling that P and Q commute with Σ_i, $i = 1, 2, 3$, and thus with the V_i as well; therefore, $\mathcal{L}[\hat\rho] = 0$, for any $\rho(0)$, as a consequence of $\mathcal{L}[\hat\rho_0] = 0$.

The problem of finding all invariant states of the dynamics (7) is then reduced to that of identifying a stationary state $\hat\rho_0$ with all eigenvalues nonzero. Although in principle this amounts in solving a linear algebraic equation, in practice it can be rather difficult for general master equations of the form (1). Nevertheless, in the case at hand, the problem can be explicitly solved, yielding:

$$\hat\rho_0 = \frac{1}{4}\Big[\mathbf{1}\otimes\mathbf{1} + M\,\Sigma_3 - N\,\big(S_{11} - S_{22}\big) + R\,S_{33}\Big]\,, \tag{16}$$

with

$$M = \frac{2\,B}{\lambda_1+\lambda_2}\ , \tag{17}$$

$$N = \frac{(\lambda_1-\lambda_2)\,B^2}{2(\lambda_1+\lambda_2)(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)}\ , \tag{18}$$

$$R = \frac{(\lambda_1+\lambda_2+4\,\lambda_3)\,B^2}{2(\lambda_1+\lambda_2)(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)}\ . \tag{19}$$

Note that the previously mentioned positivity conditions on the parameters λ_i, $i = 1, 2, 3$ and B, put restrictions on the components of $\hat\rho_0$ given above; in particular, one has: $0 \le 2\,R \le 1$, $M^2 \le 2R$, $M^2 + 4N^2 \le 1$, and the upper limits can be reached only when $\lambda_1 = \lambda_2$.[f]

Inserting this result in the expression (15) allows deriving the expression of the set of all equilibrium states of the dynamics (7); as expected, they take the

[e] One easily checks that P is the projection operator on one of the maximally entangled Bell states.
[f] These inequalities also guarantee that $\hat\rho_0$ as given in (16) is indeed a state, *i.e.* that all its eigenvalues are non-negative.

symmetric form of (12), with the nonvanishing components given by:

$$\hat{\rho}_3 = \frac{3+\tau}{3+2R}\, M \ , \tag{20}$$

$$\hat{\rho}_{11} = \frac{(1+2N)\tau + 2(3N-R)}{2(3+2R)} \ , \tag{21}$$

$$\hat{\rho}_{22} = \frac{(1-2N)\tau - 2(3N+R)}{2(3+2R)} \ , \tag{22}$$

$$\hat{\rho}_{33} = \frac{4R + (1+2R)\tau}{2(3+2R)} \ . \tag{23}$$

These stationary density matrices depend on the initial condition $\rho(0)$ only through the value of the parameter τ, that as already mentioned is a constant of motion for the dynamics in (7).

Now that we have completely classified the stationary states, one can study their properties, in particular with respect to quantum correlations. It turns out that $\hat{\rho}$ is in general entangled.

To explicitly show this, one can as before act with the operation of partial transposition on $\hat{\rho}$ to see whether negative eigenvalues are present. Alternatively, one can resort to one of the available entanglement measures and concurrence appears here to be the more appropriate: its value $\mathcal{C}[\rho]$ ranges from zero, for separable states, to one, for fully entangled states.[25–27] In the case of the state $\hat{\rho}$ above, one finds

$$\mathcal{C}[\hat{\rho}] = \max\left\{ \frac{(2+\Delta)}{2(3+2R)} \left[\frac{4R-3\Delta}{2+\Delta} - \tau\right],\ 0 \right\} \ , \tag{24}$$

where

$$\Delta = \left[\left(1-2R\right)^2 + 4(2R - M^2)\right]^{1/2} \ . \tag{25}$$

The expression in (24) is indeed nonvanishing, provided we start with an initial state $\rho(0)$ for which

$$\tau < \frac{4R-3\Delta}{2+\Delta} \ . \tag{26}$$

The concurrence depends linearly on the initial parameter τ; it assumes its maximum value $\mathcal{C}[\hat{\rho}] = 1$ when $\tau = -3$, as for the state P in (14), and reaches zero at $\tau = (4R - 3\Delta)/(2+\Delta) \leq 1$.

This result is remarkable, since it implies that the dynamics in (7) not only can initially generate entanglement: it can continue to enhance it even in the asymptotic long time regime. In other terms, prepare the two atoms in a separable state at $t = 0$; then, provided the condition (26) is satisfied, their long time equilibrium state will turn out to be entangled.

Entanglement enhancement is not limited though to initially separable states: one can easily check that the phenomenon of entanglement production through a purely noise mechanism takes place also when the initial state $\rho(0)$ already has a

non-vanishing concurrence. As an example, let us consider the following initial state, built out of the two projector operators introduced in (14):

$$\rho(0) = \frac{s}{3} Q + (1-s) P \ ; \tag{27}$$

it interpolates between the completely mixed (separable) state obtained for $s = 3/4$ and the totally entangled state P. Provided $s < 1/2$, this state is entangled, with $\mathcal{C}[\rho(0)] = 1 - 2s$. The difference in concurrence as this initial state evolves to its corresponding asymptotic one $\hat{\rho}$ turns out to be

$$\mathcal{C}[\hat{\rho}] - \mathcal{C}[\rho(0)] = 2s \left[1 - \frac{2+\Delta}{3+2R} \right] , \tag{28}$$

which is indeed non vanishing. As a final remark, notice that this enhancement in concurrence vanishes as s approaches zero; in other terms, the maximally entangled state P can never be reached as an asymptotic state unless one already starts with it at $t = 0$: P results an isolated fixed point of the dynamics generated by (7).

Appendix

We collect here the algebraic relations obeyed by the nine hermitian, traceless matrices $\Sigma_i = \sigma_i \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_i$ and $S_{ij} = \sigma_i \otimes \sigma_j + \sigma_j \otimes \sigma_i$, $i, j = 1, 2, 3$, introduced in (8) and (9). As mentioned in the text, altogether they form a closed algebra under matrix multiplication. In fact, using $\sigma_i \sigma_j = \delta_{ij} + i \sum_{k=1}^{3} \varepsilon_{ijk}\, \sigma_k$, a direct computation yields:

$$\begin{aligned}
\Sigma_i\, \Sigma_j &= 2\, \delta_{ij}\, \mathbf{1} \otimes \mathbf{1} + i \sum_{k=1}^{3} \varepsilon_{ijk}\, \Sigma_k + S_{ij} \ , \\
S_{ij}\, \Sigma_k &= \delta_{ik}\, \Sigma_j + \delta_{jk}\, \Sigma_i + i \sum_{l=1}^{3} \varepsilon_{ikl}\, S_{lj} + i \sum_{l=1}^{3} \varepsilon_{jkl}\, S_{il} \ , \\
\Sigma_k\, S_{ij} &= \delta_{ik}\, \Sigma_j + \delta_{jk}\, \Sigma_i - i \sum_{l=1}^{3} \varepsilon_{ikl}\, S_{lj} - i \sum_{l=1}^{3} \varepsilon_{jkl}\, S_{il} \ , \\
S_{ij}\, S_{kl} &= 2\big(\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk}\big) \mathbf{1} \otimes \mathbf{1} + i \sum_{r=1}^{3} \Big(\delta_{ik} \varepsilon_{jlr} + \delta_{jk} \varepsilon_{ilr} + \delta_{il} \varepsilon_{jkr} + \delta_{jl} \varepsilon_{ikr} \Big) \Sigma_r \\
&\qquad - \Big(2\, \delta_{ij}\, \delta_{kl} - \delta_{ik}\, \delta_{jl} - \delta_{il}\, \delta_{jk} \Big)\, S + 2 \Big(\delta_{ij}\, S_{kl} + \delta_{kl}\, S_{ij} \Big) \\
&\qquad - \delta_{ik}\, S_{jl} - \delta_{il}\, S_{jk} - \delta_{jk}\, S_{il} - \delta_{jl}\, S_{ik} \ ,
\end{aligned}$$

where $S = \sum_{r=i}^{3} S_{ii}$. From these relations, one immediately sees that the commutant of the linear span of the set $\{\Sigma_i :\ i, j = 1, 2, 3\}$ contains two elements, $\mathbf{1} \otimes \mathbf{1}$ and S. As explained in Section 3, this result allows classifying all stationary states of the open dynamics generated by the evolution equations in (7).

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